# Re-expansions on compact Lie groups


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Abstract. In this paper we refine the re-expansion problems for the one-dimensional torus and extend them to the multidimensional tori and to compact Lie groups. First, we establish weighted versions of classical re-expansion results in the setting of multi-dimensional tori. A natural extension of the classical re-expansion problem to general compact Lie groups can be formulated as follows: given a function on the maximal torus of a compact Lie group, what conditions on its (toroidal) Fourier coefficients are sufficient in order to have that the group Fourier coefficients of its central extension are summable. We derive the necessary and sufficient conditions for the above property to hold in terms of the root system of the group. Consequently, we show how this problem leads to the re-expansions of even/odd functions on compact Lie groups, giving a necessary and sufficient condition in terms of the discrete Hilbert transform and the root system. In the model case of the group SU(2) a simple sufficient condition is given.



2010 *Mathematics Subject Classification.* Primary: 43A30: Secondary: 43A50, 43A75, 42A20, 42B05.

*Key words and phrases.* Fourier series, re-expansion, compact Lie groups, Hilbert transform.

The authors were supported in parts by the FWO Odysseus Project G.0H94.18N: Analysis and Partial Differential Equations, EPSRC grant EP/R003025/1 and by the Leverhulme Grant RPG-2017-151.

## 1. Introduction

In the 50-s (see, e.g., [IT55] or in more detail [Kah70, Chapters II and VI]), the following problem in Fourier Analysis attracted much attention:

*Let $\{a_k\}_{k=0}^{\infty}$ be the sequence of the Fourier coefficients of the absolutely convergent sine (cosine) Fourier series of a function $f : \mathbb{T} = [-\pi, \pi) \to \mathbb{C}$, that is $\sum |a_k| < \infty$. Under which conditions on $\{a_k\}$ the re-expansion of $f(t)$ ($f(t) - f(0)$, respectively) in the cosine (sine) Fourier series will also be absolutely convergent?*

In general, the answer is negative and re-expansion not always leads to an absolutely convergent series. In this paper we shall present the necessary and sufficient condition when the positive answer is the case, while in earlier works only a sufficient condition was given, sharp on the whole class. It is quite simple and is the same in both cases:

$$\sum_{k=1}^{\infty} |a_k| \ln(k+1) < \infty. \tag{1.1}$$

In [Lif14], a similar problem of the integrability of the re-expansion for Fourier transforms of functions defined on $\mathbb{R}_+ = [0, \infty)$ has been studied. Surprisingly, necessary and sufficient conditions in terms of the membership of the sine or cosine Fourier transform in a certain Hardy space have been found.

In the present paper, we consider a similar problem on compact Lie groups. There are special features in this study. A natural extension of the classical re-expansion problem on a general compact Lie group $G$ can be formulated as a two-fold problem: first, given a function $f = \sum\limits_{k\in\mathbb{Z}^l} a_k e^{ikt}$ on the maximal torus of $G$, what conditions on its Fourier coefficients $\{a_k\}_{k\in\mathbb{Z}^l}$ are sufficient in order to have that the group Fourier coefficients of its central extension are summable (namely, to have $\widehat{\mathrm{Ext}[f]} \in \ell^1_{sch}(\widehat{G})$, with the appearing symbols explained in the following sections). The necessary and sufficient condition for this will be given in Theorem 3.2 in terms of the root structure of the group. Second, we can consider the following procedure: take a function on a group, restrict it to the maximal torus, re-expand, and then extend to the group, asking the question of the summability of the new series. Consequently, in Theorem 3.6 we show how this problem leads to the re-expansions of even/odd functions on compact Lie groups, and we derive a necessary and sufficient condition for it. In Theorem 3.7 we demonstrate the obtained criterion on the case of the compact Lie group SU(2).

The outline of the paper is as follows. In Section 2 we improve the old known results in dimension one by obtaining necessary and sufficient conditions rather than just (1.1) and further extend them to the multivariate case and both to the weighted case. In the next Section 3 we present general results for compact Lie groups $G$ and a special case $G = \mathrm{SU}(2)$. Then we give detailed proofs.

## 2. Absolutely convergent Fourier series

In this section we refine (1.1) in the sense that, like for the Fourier transforms, the necessary and sufficient conditions will be established. Analogously to the case of Fourier transform, they will be given in terms of the integrability of Hilbert transforms, but discrete. The background for this analysis will be given in the next subsection. After proving one-dimensional results, we will give their multivariate extensions. However, for generalisations to compact Lie groups it is more representative to deal with the weighted absolute convergence. We will present corresponding estimates both in dimension one and in several dimensions.

2.1. **Discrete Hilbert transforms.** For the sequence $a = \{a_k\} \in \ell^1$, $\ell^1 := \ell^1(\mathbb{Z})$, the discrete Hilbert transform is defined for $n \in \mathbb{Z}$ as (see, e.g., [Kin09, (13.127)])

$$\hbar a(n) = \sum_{\substack{k=-\infty \\ k \neq n}}^{\infty} \frac{a_k}{n-k}. \tag{2.1}$$

If the sequence $a$ is either even or odd, it suffices to consider the sequence only on $\mathbb{Z}_+$, whereas the corresponding Hilbert transforms $\hbar^e$ and $\hbar^o$ may be expressed in a special form (see, e.g., [And77] or [Kin09, (13.130) and (13.131)]). We hope that keeping the same notation $a$ will not lead to any confusion. More precisely, if $a$ is even, with $a_0 = 0$, we have $\hbar^e(0) = 0$ and we define

$$\hbar^e a(n) = \sum_{\substack{k=1 \\ k \neq n}}^{\infty} \frac{2na_k}{n^2 - k^2} + \frac{a_n}{2n}, \quad n = 1, 2, \ldots. \tag{2.2}$$

If $a$ is odd, with $a_0 = 0$, we define

$$\hbar^o a(n) = \sum_{\substack{k=1 \\ k \neq n}}^{\infty} \frac{2ka_k}{n^2 - k^2} - \frac{a_n}{2n}, \quad n = 0, 1, 2, \ldots. \tag{2.3}$$

Of course, $\frac{a_0}{0}$ is considered to be zero.

One of the best sources for the theory of discrete Hilbert transforms and discrete Hardy spaces is [BC98]. For weighted estimates for them, see [And77], [ST11a], [ST11b] and [Lif16b].

2.2. **One-dimensional case.** The obtained condition is quite simple and is the same in both cases:

$$\sum_{k=1}^{\infty} |a_k| \ln(k+1) < \infty. \tag{2.4}$$

Analyzing the proof, say, in [IT55], one can see that in fact more general results are hidden in the proofs. They can be given in terms of the (discrete) Hilbert transform.

**Theorem 2.1.** *In order than the re-expansion $\sum b_k \sin kt$ of $f(t) - f(0)$ with the absolutely convergent cosine Fourier series be absolutely convergent, it is necessary and sufficient that the discrete Hilbert transform $\hbar a$ of the sequence $a$ of the sine Fourier coefficients of $f$ is summable.*

*Similarly, in order than the re-expansion* $\sum b_k \cos kt$ *of* $f$ *with the absolutely convergent sine Fourier series be absolutely convergent, it is necessary and sufficient that the discrete Hilbert transform* $\hbar a$ *of the sequence* $a$ *of the cosine Fourier coefficients of* $f$ *is summable.*

The proofs can be started along the same lines as in the mentioned sources, first of all in [IT55]. For the first part of Theorem 2.1, [IT55] contains the fact that, for $b = \{b_k\}_{k=0}^{\infty}$,

$$b = \hbar_-^e a \tag{2.5}$$

while for the second part of Theorem 2.1, there similarly holds

$$b = \hbar_-^o a, \tag{2.6}$$

where

$$b = \hbar_-^e a = \frac{2}{\pi} \sum_{\substack{k=1 \\ k-n\, odd}}^{\infty} a_k \left( \frac{1}{n+k} + \frac{1}{n-k} \right) = \frac{4}{\pi} \sum_{\substack{k=1 \\ k-n\, odd}}^{\infty} \frac{n a_k}{n^2 - k^2} \tag{2.7}$$

and

$$b = \hbar_-^o a = \frac{2}{\pi} \sum_{\substack{k=1 \\ k-n\, odd}}^{\infty} a_k \left( \frac{1}{n+k} + \frac{1}{k-n} \right) = \frac{4}{\pi} \sum_{\substack{k=1 \\ k-n\, odd}}^{\infty} \frac{k a_k}{k^2 - n^2} \tag{2.8}$$

is the halved even and odd discrete Hilbert transform, respectively. Indeed, for $f : [0, \pi] \to \mathbb{C}$, with $f(0) = 0$ for simplicity (which is necessary as well as $f(\pi) = 0$), in the first case

$$f_e(t) = \sum_{k=1}^{\infty} a_k \cos kt, \tag{2.9}$$

with $a \in \ell^1$, while

$$f_s(t) = \sum_{n=1}^{\infty} b_n \sin nt, \tag{2.10}$$

with

$$
\begin{aligned}
b_n &= \frac{2}{\pi}\int_0^\pi f(t)\sin nt\,dt = \frac{2}{\pi}\sum_{k=1}^\infty a_k \int_0^\pi \cos kt \sin nt\,dt \\
&= \frac{1}{\pi}\sum_{k=1}^\infty a_k \int_0^\pi [\sin(n+k)t + \sin(n-k)t]\,dt \\
&= \frac{2}{\pi}\sum_{\substack{k=1\\ k-n\, odd}}^\infty a_k \left(\frac{1}{n+k} + \frac{1}{n-k}\right).
\end{aligned}
$$

We remark that the notations $f_e$ and $f_o$ is used since both equal $f$ on the half-interval.

As above, in the second case

$$
f_s(t) = \sum_{k=1}^\infty a_k \sin kt, \tag{2.11}
$$

with $a \in \ell^1$, while

$$
f_e(t) = \sum_{n=1}^\infty b_n \cos nt, \tag{2.12}
$$

with

$$
\begin{aligned}
b_n &= \frac{2}{\pi}\int_0^\pi f(t)\cos nt\,dt = \frac{2}{\pi}\sum_{k=1}^\infty a_k \int_0^\pi \cos nt \sin kt\,dt \\
&= \frac{1}{\pi}\sum_{k=1}^\infty a_k \int_0^\pi [\sin(n+k)t + \sin(k-n)t]\,dt \\
&= \frac{2}{\pi}\sum_{\substack{k=1\\ k-n\, odd}}^\infty a_k \left(\frac{1}{n+k} + \frac{1}{k-n}\right).
\end{aligned}
$$

This can be continued as follows:

$$\sum_{n=1}^{\infty}\left|\sum_{\substack{k=1\\ k-n\, odd}}^{\infty} a_k\left(\frac{1}{n+k}+\frac{1}{k-n}\right)\right| = \sum_{n=1}^{\infty}\Bigg|\sum_{\substack{k=1\\ k-n\, odd}}^{\infty} a_k\left(\frac{1}{n+k}+\frac{1}{k-n}\right)$$
$$+\frac{1}{2}\sum_{\substack{k=1\\ k-n\, odd, k\neq n+1}}^{\infty} a_k\left(\frac{1}{n+1+k}+\frac{1}{k-n-1}\right)$$
$$-\frac{1}{2}\sum_{\substack{k=1\\ k-n\, odd, k\neq n+1}}^{\infty} a_k\left(\frac{1}{n+1+k}+\frac{1}{k-n-1}\right)\Bigg|$$
$$=\frac{1}{2}\sum_{n=1}^{\infty}\Bigg|\sum_{\substack{k=1\\ k\neq n}}^{\infty} a_k\left(\frac{1}{n+k}+\frac{1}{k-n}\right)+\sum_{\substack{k=1\\ k-n\, odd}}^{\infty} a_k\left(\frac{1}{n+k}-\frac{1}{k+n+1}\right)$$
$$+\sum_{\substack{k=1\\ k-n\, odd}}^{\infty} a_k\left(\frac{1}{k-n}-\frac{1}{k-n-1}\right)\Bigg|. \tag{2.13}$$

Since

$$\sum_{n=1}^{\infty}\left|\sum_{\substack{k=1\\ k-n\, odd}}^{\infty} a_k\left(\frac{1}{k-n}-\frac{1}{k-n-1}\right)\right| \le C\sum_{k=1}^{\infty}|a_k|,$$

and the same is for the preceding sum, we have

$$\|\hbar_-^o a\|_{\ell^1} = \frac{1}{2}\|\hbar^o a\|_{\ell^1} + O(1). \tag{2.14}$$

In the completely similar way one can prove that

$$\|\hbar_-^e a\|_{\ell^1} = \frac{1}{2}\|\hbar^e a\|_{\ell^1} + O(1). \tag{2.15}$$

Hence, replacing the necessary and sufficient condition of the summability of the halved Hilbert transforms, which follows immediately from (2.5) and (2.6), with the summability of the usual discrete Hilbert transform, one arrives at the proof of the theorem.

In this case (2.4) is just a sufficient condition for the summability of the discrete Hilbert transform, though sharp on the whole class.

2.3. **Multidimensional case.** Let $\eta = (\eta_1, ..., \eta_d)$ denote a $d$-dimensional vector with $\eta_j = 0$ or $\eta_j = 1$ only, and $|\eta| = \eta_1 + ... + \eta_d \le d$. Of course, the vectors $\mathbf{0} = (\mathbf{0}, \mathbf{0}, ..., \mathbf{0})$ and $\mathbf{1} = (\mathbf{1}, \mathbf{1}, ..., \mathbf{1})$ are among such vectors. The vectors $\chi$ and $\zeta$ will be understood and used similarly.

Starting from a function $f : \mathbb{T}_+^d = [0,\pi]^d \to \mathbb{C}$, with $f(x_1, ..., x_{j-1}, 0, x_{j+1}, ..., x_d) = 0$, $j = 1, 2, ..., d$, we consider

$$f_\eta(x) = \sum_{k\in\mathbb{Z}_+^d} a_k(\prod_{j:\eta_j=1}\cos k_jx_j)(\prod_{j:\eta_j=0}\sin k_jx_j), \tag{2.16}$$

with $a\in\ell^1$, where now $\ell^1 := \ell^1(\mathbb{Z}_+^d)$. The problem is under what conditions in the re-expansion

$$f_{\mathbf{1}-\eta}(x) = \sum_{m\in\mathbb{Z}_+^d} b_m(\prod_{j:\eta_j=0}\cos m_jx_j)(\prod_{j:\eta_j=1}\sin m_jx_j)$$

we have $b\in\ell^1$. Since

$$\begin{aligned} b_m &= \frac{2^d}{\pi^d}\int_{\mathbb{T}_+^d} f(x)(\prod_{j:\eta_j=0}\cos m_jx_j)(\prod_{j:\eta_j=1}\sin m_jx_j)\,dx \qquad (2.17)\\ &= \frac{2^d}{\pi^d}\sum_{k\in\mathbb{Z}_+^d} a_k\int_{\mathbb{T}_+^d}(\prod_{j:\eta_j=1}\cos k_jx_j\sin m_jx_j)(\prod_{j:\eta_j=0}\sin k_jx_j\cos m_jx_j)\,dx\\ &= \frac{2^d}{\pi^d}\sum_{\substack{k\in\mathbb{Z}_+^d\\ k_j-m_j\ odd,\, j=1,2,\ldots,d}} a_k\prod_{j:\eta_j=1}\left(\frac{1}{m_j+k_j}+\frac{1}{m_j-k_j}\right)\prod_{j:\eta_j=0}\left(\frac{1}{m_j+k_j}+\frac{1}{k_j-m_j}\right). \end{aligned}$$

It is quite natural to denote the right-hand side of (2.17) by $\hbar_\eta^- a$

$$\begin{aligned} &\hbar_\eta^- a \qquad (2.18)\\ &= \sum_{\substack{k\in\mathbb{Z}_+^d\\ k_j-m_j\ odd,\, j=1,2,\ldots,d}} a_k\prod_{j:\eta_j=1}\left(\frac{1}{m_j+k_j}+\frac{1}{m_j-k_j}\right)\prod_{j:\eta_j=0}\left(\frac{1}{m_j+k_j}+\frac{1}{k_j-m_j}\right). \qquad (2.19) \end{aligned}$$

The summability of $\hbar_\eta^- a$ is the necessary and sufficient condition for $b\in\ell^1$. This is a direct multidimensional generalisation of the one-dimensional necessary and sufficient conditions of the summability of (2.5) and (2.6).

Further, applying (2.13) in each variable, we arrive at an analog of (2.14) and (2.15):

$$\|\hbar_\eta^- a\|_{\ell^1} = \frac{1}{2}\|\hbar_\eta^e\hbar_{\mathbf{1}-\eta}^o a\|_{\ell^1} + O\left(\sum_{0\le|\chi|+|\zeta|<|\eta|}\|\hbar_\chi^e\hbar_\zeta^o a\|_{\ell^1}\right), \tag{2.20}$$

where

$$\hbar_\chi^e a = (\prod_{j:\chi_j=1}\hbar_j^e)a, \qquad \hbar_\zeta^o a = (\prod_{j:\zeta_j=1}\hbar_j^o)a,$$

with $\hbar_j^e$ and $\hbar_j^o$ being $\hbar^e$ and $\hbar^o$, respectively, applied to the $j$th component of $a$. In other words, the operators on the right-hand side of (2.20) are mixed discrete Hilbert transforms. Their integrability leads, correspondingly, to the hybrid discrete Hardy spaces (see [Lif16a]).

Thus, the right-hand side of (2.20) consists of the leading term (repeated discrete Hilbert transforms applied to EACH of the components of $a$) and the remainder term that consists of the $\ell^1$ norms of the repeated Hilbert transforms applied only to a proper part of the components of $a$. Of course, $\|a\|_{\ell^1}$ is among them.

Therefore, $\|\hbar_\eta^e \hbar_{\mathbf{1}-\eta}^o a\|_{\ell^1} < \infty$ is a necessary and sufficient condition for $b \in \ell^1$ provided the remainder term in (2.20) is finite.

And, of course, applying (2.4) in each variable, we have a sufficient condition

$$\sum_{k\in\mathbb{Z}_+^d} |a_k| \prod_{j=1}^d \ln(k_j+1) < \infty. \tag{2.21}$$

2.4. **One-dimensional weighted case.** A more general problem arises if one assumes $\{k^q a_k\}$, $q = 1, 2, ...,$ belongs to $\ell^1$ and figures out when $\{n^q b_n\}$ belongs to $\ell^1$. In fact, this means that if we start with (2.9), the question reduces to that about

$$f_e^{(q)}(t) = \sum_{k=1}^\infty k^q a_k \cos(kt + \frac{q\pi}{2}), \tag{2.22}$$

and

$$(f_e^{(q)})_s(t) = \sum_{n=1}^\infty n^q b_n \sin(nt + \frac{q\pi}{2}), \tag{2.23}$$

while if we start with (2.11), the question reduces to that about

$$f_s^{(q)}(t) = \sum_{k=1}^\infty k^q a_k \sin(kt + \frac{q\pi}{2}), \tag{2.24}$$

and

$$(f_s^{(q)})_e(t) = \sum_{n=1}^\infty n^q b_n \cos(nt + \frac{q\pi}{2}). \tag{2.25}$$

The problem, in fact, reduces to the initial one if one observes that

$$(f_e^{(q)})_s(t) = \pm f_s^{(q)}(t) \tag{2.26}$$

and

$$(f_s^{(q)})_e(t) = \pm f_e^{(q)}(t). \tag{2.27}$$

This follows from integration by parts $q$ times in the formula for the Fourier coefficients, where the integrated terms vanish sometimes automatically or by assuming $f^{(j)})_s(0)$ and $f^{(j)}(\pi) = 0$, $j = 0, 1, ..., q-1$. Taking now into account the arguments of the previous subsection leads to the following assertion. To present it, we denote $a^q = \{A_k^q\} = \{k^q a_k\}$ and $b^q = \{B_k^q\} = \{k^q b_k\}$. Since we study the $\ell^1$ summability of these sequences, no matter if $-a^q$ is taken instead, or similarly $-b^q$. Also, since $q$ is integer, $\cos(kt + \frac{q\pi}{2})$ is either $\cos kt$ or $\sin kt$, while $\sin(kt + \frac{q\pi}{2})$ is, correspondingly $\sin kt$ or $\cos kt$, each time up to a sign $\pm$.

**Theorem 2.2.** *Let $f^{(j)}(0) = f^{(j)}(\pi) = 0$, $j = 0, 1, ..., q-1$. In order than the re-expansion $\sum B_k \sin kt$ of $f^{(q)}(t)$ with the absolutely convergent cosine Fourier series with coefficients $a^q$ be absolutely convergent, it is necessary and sufficient that the discrete Hilbert transform $\hbar^e a^q$ of the sequence $a^q$ is summable.*

*Similarly, in order than the re-expansion $\sum B_k \cos kt$ of $f^{(q)}$ with the absolutely convergent sine Fourier series with coefficients $a^q$ be absolutely convergent, it is necessary and sufficient that the discrete Hilbert transform $\hbar^o a^q$ of the sequence $a^q$ is summable.*

And in both cases the (sharp) sufficient condition is

$$\sum_{k=1}^{\infty} k^q |a_k| \ln(k+1) < \infty. \tag{2.28}$$

2.5. **Multidimensional weighted case.** We will generalise the results from the previous subsection to several dimensions. Let now $q = (q_1, ..., q_d)$ be a vector with integer $q_j \ge 0$. For $k \in \mathbb{Z}_+^d$,

$$k^q = k_1^{q_1} ... k_d^{q_d}.$$

Our starting assumption will now be $k^q a_k \in \ell^1(\mathbb{Z}_+^d)$.

In what follows $D^q f$ will mean the partial derivative

$$D^q f(x) = \left( \prod_{j=1}^{d} \frac{\partial^{q_j}}{\partial x_j^{q_j}} \right) f(x).$$

Starting from a function $f : \mathbb{T}_+^d = [0, \pi]^d \to \mathbb{C}$, with $f(x_1, ..., x_{j-1}, 0, x_{j+1}, ..., x_d) = 0$, $j = 1, 2, ..., d$, we consider

$$D^q f_\eta(x) = \sum_{k \in \mathbb{Z}_+^d} k^q a_k \Big( \prod_{j:\eta_j=1} \cos(k_j x_j + \frac{q_j \pi}{2}) \Big) \Big( \prod_{j:\eta_j=0} \sin(k_j x_j + \frac{q_j \pi}{2}) \Big),$$

with $k^q a \in \ell^1$, where now $\ell^1 := \ell^1(\mathbb{Z}_+^d)$. The problem is under what conditions in the re-expansion of $f$ in the form

$$(D^q f_\eta)_{\mathbf{1}-\eta}(x) = \sum_{m \in \mathbb{Z}_+^d} m^q b_m \Big( \prod_{j:\eta_j=0} \cos(m_j x_j + \frac{q_j \pi}{2}) \Big) \Big( \prod_{j:\eta_j=1} \sin(m_j x_j + \frac{q_j \pi}{2}) \Big)$$

we have $m^q b \in \ell^1$.

The proof of the next result is just a superposition and combination of the arguments from the two previous sections. As above, we can think on $\cos k_j x_j$ and $\sin k_j x_j$ instead of $\cos(k_j x_j + \frac{q_j \pi}{2})$ and $\sin(k_j x_j + \frac{q_j \pi}{2})$.

**Theorem 2.3.** *Let $D^s f(x_1, ... x_{j-1}, 0, x_{j+1}, ..., x_d) = D^s f(x_1, ... x_{j-1}, \pi, x_{j+1}, ..., x_d) = 0$, $j = 0, 1, ..., q_j - 1$, for any $s = (s_1, ..., s_d)$ with $s_j = 0, 1, ..., q_j - 1$. In order that the re-expansion*

$$\sum_{m\in\mathbb{Z}^d} m^q b_m\Big(\prod_{j:\eta_j=0}\cos m_j x_j\Big)\Big(\prod_{j:\eta_j=1}\sin m_j x_j\Big)$$

*of $D^q f$ with the absolutely convergent Fourier series with coefficients $k^q a$ is absolutely convergent, it is necessary and sufficient that the discrete Hilbert transform $\hbar^e_\eta \hbar^o_{\mathbf{1}-\eta}$ of the sequence $k^q a$ is summable provided*

$$\sum_{0\le|\chi|+|\zeta|<|\eta|} \|\hbar^e_\chi \hbar^o_\zeta k^q a\|_{\ell^1}$$

*is finite.*

And, of course, applying (2.28) in each variable, we have a sufficient condition

$$\sum_{k\in\mathbb{Z}^d_+} k^q |a_k| \prod_{j=1}^d \ln(k_j+1) < \infty. \tag{2.29}$$

## 3. Main results

Let $G$ be a compact connected simply connected Lie group of rank $l$ and denote by $\widehat{G}$ its unitary dual, i.e. the set of all irreducible inequivalent representations $\pi$. There is a one-to-one correspondence between $\widehat{G}$ and the set $\Lambda$ of the highest weights

$$\widehat{G} \ni \pi \leftrightarrow \mu = (\mu_1, \dots, \mu_l) \in \Lambda. \tag{3.1}$$

Therefore, we use the symbol $\pi$ to denote both the class of equivalent representations and the corresponding highest weight $\mu$, i.e.

$$\widehat{G} \ni \pi = \{\xi \in \widehat{G} \colon \xi \text{ is equivalent to } \pi\} = (\pi_1, \dots, \pi_l) \in \Lambda, \tag{3.2}$$

where $\pi_k = \mu_k$, $k = 1, \dots, l$. Each irreducible representation $\pi \in \widehat{G}$ has the corresponding highest weight $\mu = (\mu_1, \dots, \mu_l)$. The Killing form restricted to the Lie algebra $\mathfrak{t}$ of the maximal torus allows us to define isometric isomorphism between the representation weights and the elements of $\mathfrak{t}$.

We not briefly recall some well-known concept in the theory of compact topological groups. These can be found in many sources, see e.g. [Feg91] for basic explanations.

Denote by $T$ the maximal abelian subgroup of $G$. We identify $T$ with the $l$-dimensional torus $\mathbb{T}^l$ where $l$ is the rank of $G$. Let $R_+$ be a system of positive roots and let $W$ denote the Weyl group. If we denote by $\chi_\pi$ the character corresponding to $\pi \in \widehat{G}$ we have the Weyl character formula

$$\chi_\pi(e^{iH}) = \frac{\sum\limits_{w\in W} \det(w) e^{i(w\cdot(\mu+\delta),H)}}{\sum\limits_{w\in W} \det(w) e^{i(w\cdot\delta,H)}}, \quad H \in \mathfrak{t}, \tag{3.3}$$

where

$$\Delta(e^H) = \sum_{w\in W} \det(w) e^{i(w\cdot\delta,H)}$$

is the Weyl function and $\delta$ denotes the half-sum of all positive roots.

In the sequel, we usually denote by $t$ elements of the torus, and since the torus is a subgroup of $G$, also the corresponding elements of $G$.

We say that a function $g$ on $G$ is *central* if it satisfies

$$g(yty^{-1}) = g(t) \tag{3.4}$$

for all $y, t \in G$. There is also the Weyl integral formula for central functions on $G$,

$$\int_G f(g)\, dg = \frac{1}{|W|} \int_{\mathbb{T}^l} f(t)|\Delta(t)|^2\, dt, \tag{3.5}$$

where $|W|$ denotes the cardinality of the Weyl group $W$(i.e. the number of elements in $W$). It can be easily shown that

$$\Delta(t) = \prod_{\alpha \in R_+} (e^{i(\alpha,H)} + e^{-i(\alpha,H)} - 2), \tag{3.6}$$

where $R_+$ are the positive roots associated with $G$. The identification between $t$ and $H$ is explicitly given in (4.5).

Let $a = \{a_k\}_{k\in\mathbb{Z}^l} \in \ell^1(\mathbb{Z}^l)$ and let

$$f = \sum_{k\in\mathbb{Z}^l} a_k e^{ikt}. \tag{3.7}$$

Let us denote by $\mathrm{Ext}[f]$ the central extension of $f$ to the group $G$, i.e.

$$\{\text{functions on } \mathbb{T}^l\} \ni f \mapsto \mathrm{Ext}[f] \in \{\text{functions on } G\},$$

where we define $\mathrm{Ext}[f]$ as follows

$$\mathrm{Ext}[f](y^{-1}ty) := f(t), t \in \mathbb{T}^l, y \in G.$$

It is clear that $\mathrm{Ext}[f]$ is a central function. Analogously, we denote by Res the restriction of a function $f$ from $G$ to $\mathbb{T}^l$

$$\{\text{functions on } G\} \ni f \mapsto \mathrm{Res}[f] \in \{\text{functions on } \mathbb{T}^l\},$$

where we define $\mathrm{Res}[f]$ as follows

$$\mathrm{Res}[f] := f\big|_{\mathbb{T}^l}.$$

Motivated by the classical case, we seek summability conditions on the group Fourier coefficients in terms of the $\ell^p$-Schatten spaces $\ell^p_{sch}(\widehat{G})$ on $\widehat{G}$. Thus, for $1 \leq p < \infty$, we define the space $\ell^p_{sch}(\widehat{G})$ of matrix valued sequences $\{\sigma = \{\sigma(\pi)\}_{\pi\in\widehat{G}}\}$ endowed with the norm

$$\|\sigma\|_{\ell^p_{sch}(\widehat{G})} = \left(\sum_{\pi\in\widehat{G}} d_\pi \|\sigma(\pi)\|^p_{S^p}\right)^{\frac{1}{p}}$$

with the obvious modification for $p = \infty$, where $S^p$ are the usual Schatten-von Neumann norms of matrices.

For a function $f \in L^1(G)$, as usual, we denote its Fourier coefficients by

$$\widehat{f}(\pi) = \int_G f(g)\pi(g)^* dg,$$

where $dg$ is the bi-invariant Haar measure on $G$. We refer to [RT10] for the necessary backgrounds for the Fourier analysis on the compact Lie groups.

**Question 3.1.** Given a function $f = \sum_{k\in\mathbb{Z}^l} a_k e^{ikt}$, what conditions on its Fourier coefficients $\{a_k\}_{k\in\mathbb{Z}^l}$ are sufficient in order to have $\widehat{\mathrm{Ext}[f]} \in \ell^1_{sch}(\widehat{G})$.

**Theorem 3.2.** *Suppose that*

$$f = \sum_{k\in\mathbb{Z}^l} a_k e^{ikt}.$$

*Then the Fourier coefficients* $\widehat{\mathrm{Ext}[f]}$ *of its extension* $\mathrm{Ext}[f]$ *belong to* $\ell^1_{sch}(\widehat{G})$ *if and only if*

$$\sum_{\pi\in\widehat{G}} d_\pi \sum_{m=1}^{d_\pi} \left| \sum_{j=1}^{v} a_j \widehat{\Delta^2}(\pi_{mm} - j) \right| < +\infty, \tag{3.8}$$

*where the number* $0 < v \le |W|$ *depend only on the Weyl group* $W$ *and* $d_\pi$ *denotes the dimension of the representation* $\pi \in \widehat{G}$

The number $v$ in (3.8) appears as follows: in view of the formula (3.6), only a finite number of the (toroidal) Fourier coefficients of $\Delta^2$ are non-zero, leading to the finite number $v$ in (3.8).

The explicit expression of $d_\pi$ is given by Weyl's dimension formula in (3.3).

**Proposition 3.3.** *Let* $f \in L^1(G)$ *and* $\pi \in \widehat{G}$. *Then we have*

$$\widehat{f}(\pi) = \widehat{f}(\pi)^*, \tag{3.9}$$

*if and only if*

$$f \text{ is real-valued}, \tag{3.10}$$

$$f(g^{-1}) = f(g), \quad g \in G. \tag{3.11}$$

**Definition 3.4.** A real-valued function $f$ on $G$ is called even or odd if

$$f(g) = f(g^{-1}), \tag{3.12}$$

or

$$f(g) = -f(g^{-1}), \tag{3.13}$$

respectively.

The following property gives a partial justification to this terminology.

**Proposition 3.5.** *A function* $f$ *is odd or even on* $G = U(n)$ *in the sense of Definition 3.4 if and only if its restriction* $f\big|_{\mathbb{T}^l}$ *to the maximal torus* $\mathbb{T}^l$ *is odd or even respectively.*

Here $n = l$.

*Proof of Proposition 3.5.* Let $f$ be even. The case of odd function is analogous. Denote $h = f\big|_{\mathbb{T}^l}$. We write

$$e^{2\pi i t} = \begin{bmatrix} e^{2\pi i t_1} & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & e^{2\pi i t_l} \end{bmatrix}.$$

Since $f$ is central and even (see Definition 3.4), we have

$$h(t) = f(e^{2\pi it}) = f(e^{-2\pi it}) = h(-t).$$

If $h$ is even, then we have

$$\begin{aligned} f(x) = f(y^{-1}e^{2\pi it}y) = f(e^{2\pi it}) = h(t) = h(-t) \\ = f(e^{-2\pi it}) = f(y^{-1}e^{-2\pi it}y) = f(x^{-1}), \quad x, y \in G. \end{aligned}$$

This completes the proof. □

We write $g_\eta = \mathrm{Res}[f]$ and $\tilde{f} = \mathrm{Ext}[g_{\mathbf{1}-\eta}]$. Here the operations with $\eta$ were defined in Section 2.3. The combination of Theorem 3.2 and Theorem 2.3 immediately yields the following theorem. When we talk about odd/even re-expansions of functions on the torus, we refer to the previous constructions.

**Theorem 3.6.** *Let $G$ be a compact connected simply connected Lie group . Let $f \in L^1(G)$. Suppose that $f$ is even and its Fourier coefficients $\widehat{f}$ are integrable over $\widehat{G}$:*

$$\widehat{f} \in \ell^1_{sch}(\widehat{G}). \tag{3.14}$$

*Then the Fourier coefficients $\widehat{\mathrm{Ext}[f_{odd}]}$ of its odd re-expansion $f_{odd}$ are integrable*

$$\widehat{\mathrm{Ext}[f_{odd}]} \in \ell^1_{sch}(\widehat{G}), \tag{3.15}$$

*if and only if*

$$\sum_{\pi\in\widehat{G}} d_\pi \sum_{m=1}^{d_\pi} \left| \hbar\left( \sum_{j=1}^{v} a_j \widehat{\Delta^2}(\pi_{mm} - j) \right) \right| \leq C \sum_{\pi\in\widehat{G}} d_\pi \sum_{m=1}^{d_\pi} \left| \sum_{j=1}^{v} a_j \widehat{\Delta^2}(\pi_{mm} - j) \right|, \tag{3.16}$$

*with the notations of Theorem 3.2.*

For the model case $G = \mathrm{SU}(2)$, we give a simple sufficient condition. Here we use the standard identification $\widehat{\mathrm{SU}(2)} \simeq \frac{1}{2}\mathbb{N}_0$, see, e.g., [Vil68] or [RT10, Section 11.6], as well as [RT13, Section 4].

**Theorem 3.7.** *Let $f \in L^1(\mathrm{SU}(2))$. Suppose that $f$ is even and its Fourier coefficients $\widehat{f}$ are integrable over $\widehat{\mathrm{SU}(2)}$:*

$$\widehat{f} \in \ell^1_{sch}(\widehat{\mathrm{SU}(2)}). \tag{3.17}$$

*Then the Fourier coefficients $\widehat{\mathrm{Ext}[f_{odd}]}$ of its odd re-expansion $f_{odd}$ are integrable*

$$\widehat{\mathrm{Ext}[f_{odd}]} \in \ell^1_{sch}(\widehat{\mathrm{SU}(2)}), \tag{3.18}$$

*provided that*

$$\sum_{l\in\frac{1}{2}\mathbb{N}_0} (2l+1)\log(2l+1)|a_{2l+1}| < +\infty. \tag{3.19}$$

## 4. Proofs

*Proof of Theorem 3.7.* The group SU(2) has a one-dimensional torus, so the toroidal re-expansion construction reduces to the basic case described earlier. It can be checked straightforwardly

$$\sum_{m=1}^{2l+1}(a_{m-2}-2a_m+a_{m+2})=a_{-1}-a_1-a_{2l}-a_{2l+1}+a_{2l+2}+a_{2l+3},$$

where without the loss of generality we can assume that $a_0=a_{-1}=0$. Thus, we get

$$\sum_{m=1}^{2l+1}(a_{m-2}-2a_m+a_{m+2})=-a_{2l}-a_{2l+1}+a_{2l+2}+a_{2l+3}. \tag{4.1}$$

Then the series

$$\sum_{l\in\frac{1}{2}\mathbb{N}_0}(2l+1)\left|\hbar\sum_{m=1}^{2l+1}(a_{m-2}-2a_m+a_{m+2})\right|$$

is convergent if the series

$$\sum_{l\in\frac{1}{2}\mathbb{N}_0}(2l+1)\left|\hbar a_{2l+1}\right| \tag{4.2}$$

is convergent. Let us denote $n=2l+1, l\in\frac{1}{2}\mathbb{N}_0$. Then we rewrite (4.2) to get

$$\sum_{n\in\mathbb{N}}n\left|\hbar a_n\right|. \tag{4.3}$$

By repeating the relevant lines of proof in [IT55, page 251], we can show that the sufficient condition is as follows

$$\sum_{n\in\mathbb{N}}n\log n\,|a_n|.$$

Indeed, by (2.5), we have

$$\begin{aligned}\sum_{n\in\mathbb{N}} n\,|\hbar a_n| &= \sum_{n\in\mathbb{N}} n \left| \sum_{\substack{k\in\mathbb{N}\\ k-n=\mathrm{odd}}} a_k \left(\frac{1}{k+n}+\frac{1}{n-k}\right)\right| \\ &= \sum_{n\in\mathbb{N}} n \left| \sum_{\substack{k\in\mathbb{N}\\ k-n=\mathrm{odd}}} a_k \left(\frac{1}{k+n}+\frac{1}{n-k}-\frac{2}{n}\right)\right| \\ &= \sum_{n\in\mathbb{N}} n \left| \sum_{\substack{k\in\mathbb{N}\\ k-n=\mathrm{odd}}} \frac{k^2 a_k}{n(n+k)(n-k)}\right| \\ &= \sum_{n\in\mathbb{N}} n \left( \sum_{\substack{k=1,\dots,n-1\\ k-n=\mathrm{odd}}} \frac{k^2|a_k|}{n(n+k)(n-k)} + \sum_{\substack{k=n,\dots,\infty\\ k-n=\mathrm{odd}}} \frac{k^2|a_k|}{n(n+k)(k-n)}\right) \\ &= M+N,\end{aligned}$$

where we used the fact that $f(0)=f(\pi)$ implies that

$$\sum_{n\in\mathbb{N}} a_n = \sum_{n\in\mathbb{N}} (-1)^n a_n = 0.$$

Then we have

$$\begin{aligned} M &= \sum_{k=1}^{n-1} k^2|a_k| \sum_{n=k}^{\infty} \frac{1}{(n+k)(n-k)} \\ &\le \sum_{k=1}^{n-1} k^2|a_k| \left( \sum_{n=k}^{2k} \frac{1}{n(n-k)} + \sum_{n=2k+1}^{\infty} \frac{2}{n^2}\right) \\ &\le C \sum_{k=1}^{n-1} k|a_k| \log(k). \end{aligned}$$

And for $N$ analogously, we get

$$\begin{aligned} N = \sum_{k=1}^{\infty} k^2|a_k| \sum_{n=1}^{k} \frac{1}{(n+k)(n-k)} &\le \sum_{k=1}^{\infty} k^2|a_k| \left( \sum_{n=1}^{[\frac{k}{2}]} \frac{1}{k-n} + \sum_{n=[\frac{k}{2}]+1}^{k-1} \frac{1}{k-n}\right) \\ &\le C \sum_{k=1}^{\infty} k|a_k|(\log k + \log k). \end{aligned}$$

This completes the proof. □

*Proof of Theorem 3.2.* Every Lie group homomorphism gives rise to a Lie algebra homomorphism. The converse is true since $G$ is simply connected. In particular, for every $\pi \in \widehat{G}$, we have

$$\pi(e^{iH}) = e^{id\pi(H)}, \tag{4.4}$$

where $H \in \mathfrak{t}$. Let $\{H_1, \dots, H_l\}$ in $H$

$$\mathfrak{t} \ni H \longleftrightarrow (t_1, \dots, t_l)\colon H = \sum_{k=1}^{l} t_k H_k. \tag{4.5}$$

By (4.4) and (4.5), we have

$$\pi(e^{i \sum\limits_{k=1}^{l} t_k H_k}) = e^{i \sum\limits_{k=1}^{l} t_k d\pi(H_k)}. \tag{4.6}$$

It can be proven that matrices $d\pi(H_k)$ can be diagonalized in the representation space of each $\pi \in \widehat{G}$

$$d\pi(H_k) = \mathrm{diag}(\mu_1, \dots, \mu_s) \tag{4.7}$$

with the same $\{\mu_1, \dots, \mu_s\} \subset \mathbb{Z}^l$ from the weight diagram. We have

$$\sum_{k=1}^{l} t_k d\pi(H_k) = \begin{pmatrix} \sum\limits_{k=1}^{l} t_k \mu_1^k & & \\ & \ddots & \\ & & \sum\limits_{k=1}^{l} t_k \mu_s^k \end{pmatrix}. \tag{4.8}$$

By definition, we have

$$\widehat{f}(\pi) = \int_G f(u)\pi(u)^* \, du. \tag{4.9}$$

Since $f$ and $\pi(u)_{mm}$ are central functions for $m = 1, \dots, d_\pi$, the application of Weyl's integral formula (3.5) yields

$$\widehat{f}(\pi)_{mm} = \frac{1}{|W|} \int\limits_{\mathbb{T}^l} f(t)\overline{\pi(t)_{mm}} \Delta^2(t) \, dt. \tag{4.10}$$

By (4.7) and (4.6), we get

$$\pi(t)_{mm} = e^{i\mu_m \cdot t}. \tag{4.11}$$

It can be easily shown that

$$\Delta(t) = \prod_{\alpha \in R_+} (e^{i(\alpha,H)} + e^{-i(\alpha,H)} - 2), \tag{4.12}$$

where $R_+$ are the positive roots associated with $G$. Thus, using expansion (3.7),(4.11) and (4.12), we obtain

$$\widehat{f}(\pi)_{mm} = \int\limits_{\mathbb{T}^l} \sum_{k \in \mathbb{Z}^l} a_k e^{2\pi i (k,t)} e^{-2\pi i (\mu_m, t)} \Delta^2(t) \, dt. \tag{4.13}$$

Since the Weyl group $W$ is finite, the last sum can be represented as follows

$$\widehat{f}(\pi)_{mm} = \sum_{k=1}^{\nu} a_k \widehat{\Delta^2}(k - \pi_{mm}). \tag{4.14}$$

By the assumption, $f$ is even function (see Definition 3.4). Then its Fourier coefficients $\widehat{f}(\pi)$ are self-adjoint operators (Proposition 3.3)

$$\widehat{f}(\pi)^* = \widehat{f}(\pi).$$

Therefore, we have

$$\|\widehat{f}(\pi)\|_{S^1(\mathcal{H}^\pi)} = \sum_{m=1}^{d_\pi} |\widehat{f}(\pi)_{mm}|. \tag{4.15}$$

By definition, we have

$$\|\widehat{f}\|_{\ell^1_{sch}(\widehat{G})} = \sum_{\pi\in\widehat{G}} d_\pi \|\widehat{f}(\pi)\|_{S^1(\mathcal{H}^\pi)} = \sum_{\pi\in\widehat{G}} d_\pi \sum_{m=1}^{d_\pi} \left| \sum_{j=1}^{v} a_j \widehat{\Delta^2}(\pi_{mm} - j) \right|. \tag{4.16}$$

This completes the proof. $\square$


## Acknowledgements

The authors thank the referees for thorough reading and valuable remarks and suggestions.


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